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A theory of random rays, based on the stochastic mechanical interpretation of the parabolic wave equation, is proposed. The relation of these rays to those of geometric acoustics is discussed. The Feynman-Kac formula is used to represent the acoustic wave field as a Wiener integral, and it is shown that this representation agrees with the Markov approximation in a simple case.

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RANDOM RAYS, GEOMETRIC ACOUSTICS, AND THE PARABOLIC WAVE EQUATION*

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I. THE PARABOLIC WAVE EQUATION

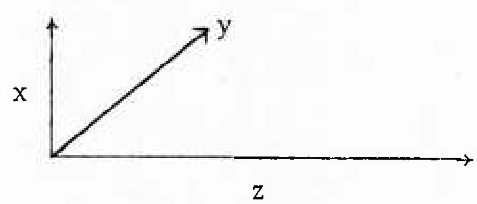
The wave equation for the complex acoustic pressure field p away from acoustic sources is

$$\nabla^2 p = \frac{1}{C^2} \frac{\partial^2 p}{\partial t^2}\ , \qquad (1)$$

where C is the speed of sound.

Here $p = p(x,y,z,t)$, $C = C(x,y,z)$, and $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

We use the coordinate system shown below, in which z is the horizontal (range) coordinate and x is the vertical coordinate:



Since p is assumed complex, the observed signal is Re(p), the actual pressure.

Following Dashen [1] and others, we assume that

$$p(x,y,z,t) = \psi(z,x,y)e^{i(kz-\omega t)}\ , \qquad (2)$$

where ψ is a complex envelope, ω is the frequency, and $k = 1/\lambda$ is the wave number, λ being wavelength.

Letting $\lambda\omega = \omega/k = C_0$, a reference sound speed, we define the index of refraction n by

$$n(x,y,z) = C_0/C(x,y,z).$$

Then (1) becomes

$$\nabla^2 p + k^2 n^2 p = 0, \qquad (3)$$

the reduced wave equation.

Putting $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and using (2), we see that (3) may be written

$$\Delta\psi + \frac{\partial^2\psi}{\partial z^2} + 2ik\frac{\partial\psi}{\partial z} + k^2(n^2-1)\psi = 0 .$$

Suppose we make the following assumptions:

A) The term $\partial^2\psi/\partial z^2$ may be neglected in comparison with $2ik\frac{\partial\psi}{\partial z}$, i.e.

$$\left|\frac{\partial^2\psi}{\partial z^2}\right| \ll \left|2ik\frac{\partial\psi}{\partial z}\right| ,$$

and

B) $\frac{1}{2}(n^2-1) \cong n-1$, i.e., $n \cong 1$.

Thus, we have

$$i\frac{\partial\psi}{\partial z} + \frac{1}{2k}\Delta\psi + k(n-1)\psi=0 ,$$

or, defining μ by

$$\mu(x,y,z) = 1-n(x,y,z), \tag{4}$$

we get

$$\boxed{\begin{gathered} i\frac{\partial\psi}{\partial z} + \frac{1}{2k}\Delta\psi - \mu k\psi = 0 , \\ \psi = \psi(z,x,y), \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} , \quad \mu = \mu(x,y,z). \end{gathered}} \tag{5}$$

Equation (5) is called the <u>parabolic wave equation</u>. As we have seen, approximating the reduced wave equation (3) by the parabolic wave equation (5) rests on the assumptions A) and B) above. Assumption B) says just that the sound speed varies little throughout the medium, i.e. the approximation $C \cong C_0$ introduces a small percentage error everywhere in the medium. This assumption is surely valid in underwater acoustics. For example, if one considers the representative speed of sound profile on p.400 of [2], one easily estimates that the approximation

$$C \cong C_{min} \cong 1475 \text{ m/sec.},$$

introduces an error of less than 5% down to a depth of 6 km.

The validity of the parabolic approximation, and of assumption A) in particular, has been carefully considered by Tappert in the survey article [3], and by Flatté et al [4]. In the latter reference, pp. 76-77, it is stated that, for typical conditions for sound propagation in the ocean,

$$\left| \frac{\partial^2 \psi}{\partial z^2} \right| / \left| k \frac{\partial \psi}{\partial z} \right| \sim 10^{-3} \quad ,$$

so that assumption A) seems quite safe in underwater acoustics. Tappert observes that the parabolic approximation applies essentially to long-range (R = range $\gg \lambda$, typically $R > 50$-60 km.) propagation in the ocean sound channel, which acts like a vertically thin, horizontally elongated waveguide; consequently if we think in terms of rays, the angle θ between the ray direction and the horizontal is small ($\theta_{max} < 16°$) for rays which are not attenuated to zero intensity over the long-range propagation. The envelope ψ varies more slowly with the range coordinate z than with depth x. The parabolic approximation retains diffraction effects, unlike geometric acoustics, which assumes that wavelengths are small enough that diffraction can be ignored (see below). The parabolic approximation is therefore valid to significantly lower frequencies than geometric acoustics. Finally, the parabolic approximation is not limited to stratified media, as are normal mode expansions.

II. GEOMETRIC ACOUSTICS APPROXIMATION TO THE PARABOLIC WAVE EQUATION

Since the parabolic wave equation retains the diffraction effects whose neglect is the essence of the geometric acoustics approximation (at least in the presence of gentle refraction, $n \cong 1$, as we have assumed - see [2], p.118), we should be able to derive geometric acoustics from the parabolic wave equation (5) via a suitable neglect of diffractive terms. We proceed

to do so, following Tappert [3, p.234 ff.].

Let us write

$$\psi(z,x,y) = A(z,x,y)e^{i\theta(z,x,y)} \quad , \tag{6}$$

where A is a real amplitude and θ is a real phase. For simplicity, here and in the sequel we ignore the (azimuthal) dependence of ψ, A, ϕ, etc. on y: we assume

$$\psi = \psi(z,x) = A(z,x)e^{i\phi(z,x)} \quad , A, \phi \text{ real} . \tag{7}$$

Now substitute (7) into (5), and divide by $e^{i\phi}$; the real part of the resulting equation yields

$$-A\frac{\partial\phi}{\partial z} + \frac{1}{2k}\left[\frac{\partial^2 A}{\partial x^2} - A\left(\frac{\partial\phi}{\partial x}\right)^2\right] - \mu kA = 0 \quad , \tag{8}$$

and the imaginary part yields

$$\frac{\partial A}{\partial z} + \frac{1}{k}\frac{\partial A}{\partial x}\frac{\partial\phi}{\partial x} + \frac{A}{2k}\frac{\partial^2\phi}{\partial x^2} = 0 \ . \tag{9}$$

Now define θ by

$$\theta(z,x) = \frac{1}{k}\frac{\partial\phi}{\partial x} \ . \tag{10}$$

Then (9) becomes

$$\frac{\partial A}{\partial z} + \theta\frac{\partial A}{\partial x} + \frac{A}{2}\frac{\partial\theta}{\partial x} = 0.$$

Multiplying this by 2A, we obtain

$$\frac{\partial}{\partial z}(A^2) + \frac{\partial}{\partial x}(\theta A^2) = 0 \ . \tag{11}$$

Also, in terms of θ, (8) is

$$-A\frac{\partial\phi}{\partial z} + \frac{1}{2k}\frac{\partial^2 A}{\partial x^2} - \tfrac{1}{2}kA\theta^2 - \mu kA = 0 \ .$$

If we differentiate this with respect to x, divide by (-kA), and use (8) again, we obtain

$$\frac{\partial\theta}{\partial z} + \theta\frac{\partial\theta}{\partial x} = \frac{-\partial\mu}{\partial x} + \frac{1}{2k^2}\frac{\partial}{\partial x}\left[\frac{1}{A}\frac{\partial^2 A}{\partial x^2}\right] . \tag{12}$$

Now the geometric acoustics approximation is that $k \gg 1$, and so the second, diffractive term on the right-hand side of (12) may be dropped:

$$\frac{\partial\theta}{\partial z} + \theta\frac{\partial\theta}{\partial x} = -\frac{\partial\mu}{\partial x} . \tag{12GE}$$

Consider the curves $x = x(z)$ defined by

$$\frac{dx}{dz} = \theta(z,x) . \tag{13}$$

Along these curves (12GE) says

$$\frac{d}{dz}\theta(z,x(z)) = -\frac{\partial\mu}{\partial x} ,$$

or by (13) again,

$$\frac{d^2x}{dz^2} = \frac{-\partial\mu}{\partial x} , \tag{14}$$

which is the differential equation for the rays of geometrical acoustics in a horizontally stratified ocean (i.e., $n=n(x)$) for which $n \cong 1$ and $\theta \ll 1$, as we assume for the parabolic approximation. Before verifying this claim, we notice that the geometrical acoustics assumption $k \gg 1$ in (10) implies that $\theta \ll 1$, so that $\tan\theta \cong \theta$; hence (13) says that θ is just the angle between a ray and the horizontal.

In general, the ray equations are (see, e.g., [5], p.79)

$$n\frac{d}{d\sigma}\left(n\frac{dx}{d\sigma}\right) = \tfrac{1}{2}\frac{\partial}{\partial x}(n^2), \tag{15}$$

assuming $n = n(x)$, where σ is arclength along a ray. Since $(d\sigma)^2=(dz)^2+(dx)^2$, and $\frac{dx}{dz} = \tan\theta$, we have

$$\left(\frac{d\sigma}{dz}\right)^2 = 1+\left(\frac{dx}{dz}\right)^2 = 1 + \tan^2\theta = \sec^2\theta ,$$

so $\frac{dz}{d\sigma} = \cos\theta$, and therefore

$$\frac{dx}{d\sigma} = \frac{dx}{dz}\cdot\frac{dz}{d\sigma} = \frac{dx}{dz}\cos\theta .$$

Thus, (15) becomes

$$n \frac{d}{d\sigma} \left[n \cos\theta \frac{dx}{dz} \right] = \tfrac{1}{2} \frac{\partial}{\partial x} (n^2) \ .$$

But Snell's law is

$$s \equiv n \cos\theta = \text{constant},$$

so the above is simply

$$ns \frac{d}{d\sigma} \left[\frac{dx}{dz} \right] = \tfrac{1}{2} \frac{\partial}{\partial x} (n^2) \ ,$$

or

$$s^2 \frac{d^2x}{dz^2} = \tfrac{1}{2} \frac{\partial}{\partial x} (n^2) \ .$$

But $n \cong 1$ and $\theta \ll 1$ imply $s \cong 1$, and $\frac{1}{2}(n^2-1) \cong n-1 = -\mu$, and so the above equation reduces to (14), as claimed above.

Before leaving the geometric acoustics approximation temporarily, notice that (11) is just a transport equation for the acoustic power A^2 along the rays.

III. RANDOM RAY MECHANICS

We are going to propose a theory of rays $z \to x(z)$ in which $x(z)$ is a random process indexed by the range coordinate z. This theory is Nelson's stochastic mechanics [e.g., 6-9], and the presentation initially follows Nelson [8]. Our goal is to recapture diffractive effects for geometric acoustics by regarding $x(z)$, for fixed z, as spread out vertically according to a probability density $\rho(x,z)$. We shall see mathematically that such a picture is equivalent to the parabolic wave equation (5). Of course, Nelson's theory is a stochastic version of the Schrödinger equation of quantum mechanics, but this equation is formally identical to (5), provided we replace the time coordinate by the range coordinate z. We shall omit most proofs in this section, referring the interested reader to [6-9].

Consider a random path, or ray, governed by the Ito stochastic differential equation [10-12]

$$dx(z) = b(x,z)dz + \sqrt{2\nu}\, dw(z), \tag{16}$$

where b is called the drift, ν is a constant called the diffusion coefficient (we retain these terms from the standard literature where z is replaced by t, although calling b the "spread" and ν the "diffraction coefficient" would perhaps be more appropriate here), and w(z) is standard Brownian motion. Let $\rho(x,z)$ be the probability density for x(z). Then ρ satisfies the equation

$$\frac{\partial\rho}{\partial z} = \nu \frac{\partial^2\rho}{\partial x^2} - \frac{\partial}{\partial x}(b\rho) , \tag{17}$$

which physicists call the Fokker-Planck equation and mathematicians call Kolmogorov's forward equation (see, e.g. [13], p.275).

We may also describe the process x(z) with the direction of z reversed (analogous to time-reversal when z is replaced by t); since the reversed process is again of the same type (cf. [13], p.83), we have

$$dx = b_*(x,z)dz + \sqrt{2\nu}\, dw_* , \tag{18}$$

for a suitable backward drift b_* and a (possibly different) Brownian motion w_*, where we understand $dx = x(z) - x(z-dz)$, $dz > 0$. We then have the backward Fokker-Planck equation

$$\frac{\partial\rho}{\partial z} = -\nu \frac{\partial^2\rho}{\partial x^2} - \frac{\partial}{\partial x}(b_*\rho) . \tag{19}$$

Let v, the current velocity, be defined by

$$v = (b+b_*)/2 . \tag{20}$$

Then (17) and (19) yield the equation of continuity,

$$\frac{\partial\rho}{\partial z} = - \frac{\partial}{\partial x}(v\rho) . \tag{21}$$

Let the osmotic velocity u be defined by

$$u = (b-b_*)/2 . \tag{22}$$

Then [7, p.105] we have

$$u = \frac{\nu}{\rho}\frac{\partial\rho}{\partial x} . \tag{23}$$

The formula (23) for Brownian motion was in essence derived by Einstein ([14], p.9 ff.); the term "osmotic velocity" is used because, in the case of Brownian motion of particles suspended in a liquid, u is the velocity particles must acquire to overcome osmotic effects ([7], p.21).

We now assume that there exists a function $V(x,z)$ such that

$$\frac{d}{dz}\int_{-\infty}^{\infty}(\tfrac{1}{2}u^2+\tfrac{1}{2}v^2 + V)\rho(x,z)dx = 0 . \tag{24}$$

In the diffusion case $(z \to t)$, V is the potential energy, and (24) expresses energy conservation: the average total energy is constant in time. We shall see that in our case we may take $V = \mu$, the variation $\mu = 1-n$ of the index of refraction from its typical value 1 (recall that $n \cong 1$ in the parabolic approximation). From (24) it follows ([8], p.174) that

$$-(\nu\frac{\partial^2 u}{\partial x^2} + u\frac{\partial u}{\partial x}) + (\frac{\partial v}{\partial z} + v\frac{\partial v}{\partial x}) + \frac{\partial V}{\partial x} = 0 \tag{25}$$

If we differentiate (23) with respect to z and then use (21), we find that

$$\frac{\partial u}{\partial z} = -\nu\frac{\partial^2 v}{\partial x^2} - \frac{\partial}{\partial x}(uv) . \tag{26}$$

Assuming that ρ never vanishes, let R be given by

$$R = \tfrac{1}{2}\log\rho, \tag{27}$$

so that (23) becomes

$$u = 2\nu\frac{\partial R}{\partial x} . \tag{28}$$

Also write v as a gradient: choose a function S such that

$$v = 2\nu\frac{\partial S}{\partial x} . \tag{29}$$

(Note: the analogues of (28) and (29) in [8] are unfortunately in error.)

Now define ψ by

$$\psi = e^{R+iS} \tag{30}$$

and choose

$$\nu = \frac{1}{2k} \ . \tag{31}$$

Then a simple calculation shows that ψ satisfies the parabolic wave equation (5):

$$i \frac{\partial \psi}{\partial z} + \frac{1}{2k} \frac{\partial^2 \psi}{\partial x^2} - \mu k \psi = 0 \ , \tag{5}$$

provided we take V to be

$$V = \mu \ . \tag{32}$$

(Actually, since we must integrate (25) and (26) with respect to x to derive (5), a seemingly arbitrary complex constant of integration $\alpha = \alpha(z)$ is introduced, so that we get

$$(*) \qquad i\frac{\partial \psi}{\partial z} + \frac{1}{2k} \frac{\partial^2 \psi}{\partial x^2} - \mu k \psi + \alpha(z)\ \psi = 0$$

But, because $\rho = \overline{\psi}\psi$ is a probability density, $\int_{-\infty}^{\infty} \rho(z,x)dx = 1$, for all z. Since (*) multiplied by $\overline{\psi}$ and its complex conjuate multiplied by ψ imply

$$i \frac{\partial \rho}{\partial z} = -\frac{1}{2k} \frac{\partial}{\partial x}\left[\overline{\psi} \frac{\partial \psi}{\partial x} - \psi \frac{\partial \overline{\psi}}{\partial x}\right] + (\overline{\alpha}-\alpha)\rho \quad ,$$

we have

$$0 = i \frac{\partial}{\partial z} \int_{-\infty}^{\infty} \rho \ dx = (\overline{\alpha}-\alpha) \int_{-\infty}^{\infty} \rho \ dx = (\overline{\alpha}-\alpha) \ ,$$

so α must be real. But S in (29) is determined only up to an arbitrary additive function of z. Choosing S appropriately, we may insure $\alpha = 0$. This argument is due to Nelson [7, p.131].)

Conversely, suppose ψ satisfies (5), and define R, S, u, v, by (28), (29), and (30). Then u and v satisfy (25) and (26), with $V = \mu$. Thus, each solution x(t) of (16) gives rise to a solution ψ of the parabolic wave equation (5), and conversely: knowing ψ, we know u and v, and hence b. In this sense, (5) is strictly equivalent to a theory of random rays governed by (16). From the point of view of random ray theory, solving (5) for ψ is merely a convenient method of determining the drift b, and hence what stochastic

differential equation the rays obey. Solving the linear equation (5) is simpler than solving directly the coupled, non-linear system (25), (26) of partial differential equations.

It will be convenient to have an alternative account of the dynamics embodied in (25). IF $f(x,z)$ is a smooth function, define the mean forward derivative $Df(x(z),z)$ of $f(x(z),z)$ by

$$Df(x(z),z) = \lim_{\Delta z \downarrow 0} E\left\{\frac{f(x(z+\Delta z),z+\Delta z)-f(x(z),z)}{\Delta z}\middle| P_z\right\}, \tag{33}$$

and the mean backward derivative $D_*f(x(z),z)$ of $f(x(z),z)$ by

$$D_*f(x(z),z) = \lim_{\Delta z \downarrow 0} \left\{\frac{f(x(z),z) - f(x(z-\Delta z),z-\Delta z)}{\Delta z}\middle| F_z\right\}, \tag{34}$$

where $P_z = \sigma\{x(s)|0 \le s \le z\}$ is the σ-field generated by $x(s)$ for $s \le z$, $F_z = \sigma\{x(0)|s \ge z\}$ is the σ-field generated by $x(s)$ for $s \ge z$, and $E\{X|F\}$ is the conditional expectation of the random variable X with respect to the σ-field F.

Now, by Ito's lemma (e.g., [11], p.33), we have

$$Df(x(z),z) = \frac{\partial f}{\partial z}(x(z),z) + b(x(z),z)\,\frac{\partial f}{\partial x}(x(z),z) + \nu\,\frac{\partial^2 f}{\partial x^2}(x(z),z), \tag{35}$$

and similarly,

$$D_*f(x(z),z) = \frac{\partial f}{\partial z}(x(z),z) + b_*(x(z),z)\,\frac{\partial f}{\partial x}(x(z),z) - \nu\,\frac{\partial^2 f}{\partial x^2}(x(z),z). \tag{36}$$

In particular, if $f(x(z),z) = x(z)$, then

$$Dx(z) = b(x(z),z) \tag{37}$$

and

$$D_*x(z) = b_*(x(z),z)\ . \tag{38}$$

Now, define the stochastic acceleration a by

$$a = \tfrac{1}{2}\{DD_*+D_*D\}(x(t)). \tag{39}$$

Then by (37) and (38),

$$a = \tfrac{1}{2}\{Db_* + D_*b\} \ ,$$

or, by (35) and (36),

$$a = \frac{\partial}{\partial z}\left(\frac{b+b_*}{2}\right) + \frac{\partial}{\partial x}\left(\frac{b\ b_*}{2}\right) + \nu \frac{\partial^2}{\partial x^2}\left(\frac{b_*-b}{2}\right) \ .$$

In terms of $u = (b-b_*)/2$ and $v = (b+b_*)/2$, $bb_* = v^2-u^2$, so this is

$$a = \frac{\partial v}{\partial z} + v\frac{\partial v}{\partial z} - u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} \ . \tag{40}$$

Hence, (25) is simply

$$a = -\frac{\partial V}{\partial x} \ , \tag{41}$$

or, defining the force F by $F = -\partial V/\partial x$,

$$F = a \ , \tag{42}$$

which is just Newton's second law for a particle of unit mass. Thus, with the stochastic acceleration defined by (39), the dynamical equation (25) is simply Newton's second law, rather than energy conservation (eq. (24)).

We also have an important formula relating the ordinary z-derivative of average quantities to averages of their mean forward and mean backward derivatives ([7], p.98; [9], p.204): if $f(x,z)$ and $g(x,z)$ are smooth functions, and E denotes averaging in the probability space of the process $x(z)$, then

$$\frac{d}{dz}E(f(x(z),z)g(x(z),z)) = E[(Df(x(z),z))(g(x(z),z))+(f(x(z),z))(D_*g(x(z),z))]. \tag{43}$$

In particular, with $g \equiv 1$ this yields

$$\frac{d}{dz}E(f(x(z),z)) = E(Df(x(z),z)) \ , \tag{44}$$

and, interchanging f and g and putting $g \equiv 1$, we find

$$\frac{d}{dz}E(f(x(z),z)) = E(D_*f(x(z),z)). \tag{45}$$

Simple consequences of (44) and (45) are

$$\frac{d}{dz} E(f(x(z),z)) = E(\tfrac{1}{2}(D+D_*)f(x(z),z)) \tag{46}$$

and

$$E((D-D_*)f(x(z),z)) = 0. \tag{47}$$

In particular, for $f(x(z),z) = x(z)$, (46) and (47) imply, by (37) and (38),

$$\frac{d}{dz} E(x(z)) = E(v(x(z),z)) \tag{48}$$

and

$$E(u(x(z),z)) = 0. \tag{49}$$

Differentiating (48) with respect to z, and using (46), we obtain

$$\frac{d^2}{dz^2} E(x(z)) = E(\tfrac{1}{2}(D+D_*)v(x(z),z)) ,$$

or, by (35) and (36),

$$\frac{d^2}{dz^2} E(x(z)) = E(\frac{\partial v}{\partial z}(x(z),z) + v(x(z),z) \frac{\partial v}{\partial x}(x(z),z)). \tag{50}$$

Now, let us take expectations in (40) to get

$$E(a(z)) = E(\frac{\partial v}{\partial z}(x(z),z) + v(x(z),z) \frac{\partial v}{\partial x}(x(z),z)) - E(u(x(z),z)\frac{\partial u}{\partial x}(x(z),z) + \nu \frac{\partial^2 u}{\partial x^2} (x(z),z)) .$$

We shall show that the average in the second term is zero. Indeed, by (47) with $f = u$,

$$0 = E(\tfrac{1}{2}(D-D_*)u(x(z),z)),$$

and by (35) and (36), this is

$$0 = E(u(x(z),z) \frac{\partial u}{\partial x}(x(z),z) + \nu \frac{\partial^2 u}{\partial x^2} (x(z),z)) ,$$

as claimed. Hence

$$E(a(z)) = E(\frac{\partial v}{\partial z}(x(z),z) + v(x(z),z)\frac{\partial v}{\partial x}(x(z),z)). \tag{51}$$

Comparing (50) and (51), we have

$$\frac{d^2}{dz^2} E(x(z)) = E(a(z)) ,$$

or, recalling (41),

$$\frac{d^2}{dz^2} E(x(z)) = -E(\frac{\partial V}{\partial x}(x(z),z)) \ . \tag{52}$$

This equation says that the average ray $z \to E(x(z))$ feels the average force, and not $\frac{-\partial V}{\partial x}(E(x(z)),z)$, the force along the average ray.

IV. RANDOM RAYS AND THE GEOMETRIC ACOUSTICS APPROXIMATION

Suppose we have a solution ψ of the parabolic wave equation (5). Thus, we have vector fields u and v, defined by (28)-(30), so we may define three different kinds of paths, or rays:

A. <u>geometric acoustics rays $x^G: z \to x^G(z)$</u>:

These are defined as solutions to the ray equation (14):

$$\frac{d^2 x^G}{dz^2}(z) = \frac{-\partial\mu}{\partial x}(x^G(z)). \tag{53}$$

B. <u>parabolic rays $x^P: z \to x^P(z)$</u>:

These are defined by (13), so that

$$\frac{dx^P}{dz} = v(x^P(z),z) \tag{54}$$

(note that θ of section II is the same as v of section III: cf. (6) and (10) with (29)-(31)), and thus, by (12), satisfy

$$\frac{d^2 x^P}{dz^2}(z) = -\frac{\partial\mu}{\partial x}(x^P(z)) + \frac{1}{2k^2}\frac{\partial}{\partial x}\left[\frac{1}{A}\frac{\partial^2 A}{\partial x^2}\right](x^P(z),z) \ ,$$

or,

$$\frac{d^2 x^P}{dz^2}(z) = -\frac{\partial\mu}{\partial x}(x^P(z)) + \frac{1}{2k}\frac{\partial^2 u}{\partial x^2}(x^P(z),z) + u(x^P(z),z)\frac{\partial u}{\partial x}(x^P(z),z) \ , \tag{55}$$

where the last equation follows from $u = \frac{1}{k}\frac{\partial R}{\partial x} = \frac{1}{k}\frac{\partial}{\partial x}(\log A)$, by direct computation, or by comparing (12) and (25).

C. random rays $x: z \to x(z)$: defined as a random process as in section III; in particular the average random ray $E(x(z))$ satisfies (52):

$$\frac{d^2}{dz^2}E(x(z)) = -E(\frac{\partial\mu}{\partial x}(x(z),z)) \ . \tag{56}$$

All three objects x^G, x^P, and $E(x(z))$ are in general distinct, since they satisfy the different equations (53), (55), and (56).

Since the parabolic wave equation is just the Schrödinger equation of quantum mechanics, and since we expect ray theory to be meaningful when $k \gg 1$, i.e., when $1/k \ll 1$, where $(1/k)$ is playing the role of $\hbar$ = Planck's constant divided by 2π in (5), we can interpret some approximate relationships among x^P, $x(z)$, and x^G in terms of approximating quantum mechanics by classical mechanics when $\hbar \ll 1$. For example, the approximation

$$x^P \cong x^G$$

achieved as in section II by dropping the last two terms on the right side of (55) has been well studied in the quantum-mechanical context. It can be understood as ignoring a "quantum-mechanical potential," thereby converting a non-linear fluid-mechanical system (the "Madelung fluid") into a linear fluid-mechanical system (the "Hamilton-Jacobi fluid") whose dynamics constitute a well-known version of classical mechanics (see, e.g., [15] and references therein).

The approximation $x(z) \cong x^G(z)$ amounts to replacing $\frac{1}{2}(DD_* + D_*D)$ by $\frac{d^2}{dt^2}$, i.e., replacing stochastic acceleration by the usual second time derivative. Studying the semiclassical limit of quantum mechanics in its stochastic mechanical version is a relatively untouched field of research (however, see the interesting paper [16]).

One might hope that $E\ x(z) = x^G$ for appropriate initial conditions, but this is false in general: the average random ray is not a geometric acoustics

ray, since they satisfy different differential equations, (53) and (56). The failure of $E(x(z))$ to be a geometric acoustics ray is measured by

$$E(\frac{\partial\mu}{\partial x}(x(z),z)) - \frac{\partial\mu}{\partial x}(E(x(z),z)) ,$$

the error made in approximating the mean force by the force on the mean ray. Estimation of the size of this error seems worth further investigation, as do the approximation $x^P \cong x^G$ and $x(z) \cong x^G$, understood as semiclassical quantum-mechanical limits.

A related question is the estimation of

$$E(x(z)) - x^P(z),$$

which might well proceed from the formula (recall that $x^P(z)$ is non-random)

$$E(x(z)-x^P(z)) = \int_{s=0}^{z} E[v(x(s),s)-v(x^P(s),s)]ds ,$$

together with the assumption that v is Lipschitz.

V. THE FEYNMAN-KAC FORMULA AND SOLUTIONS TO THE PARABOLIC WAVE EQUATION

Invented in 1948, Feynman's path integral [17, 18] has long since established itself as a powerful alternative formulation of quantum dynamics, embodying direct heuristic appeal and achieving success in an impressively broad range of applications [see, e.g., 19]. Indeed, Dashen's pioneering application of Feynman's path integral to wave propagation in random media [1] is the primary stimulus of the present work. As Dashen has shown, the Feynman integral representation of the solution ψ of the Parabolic Wave Equation with random fluctuation μ allows one to calculate moments, correlation functions, and various other important characteristics of the random field ψ.

From a mathematical point of view, however, Feynman's path integral has proved profoundly problematical. It was soon realized [20] that Feynman's integral cannot be understood as an ordinary integral, since it arises from no measure. Various attempts at rigorous formulations of Feynman integrals via generalizations of the concept of measure have proved fertile fields for mathematical research. But the most significant development for our purposes is Kac's discovery [21; 22, pp. 165-172] that, provided we begin with the nonhomogenous heat equation instead of the parabolic wave equation, Feynman's procedure does lead to a well-defined integral, the Wiener integral, arising from a well-defined measure, Wiener measure. The resulting formula for the solution of the heat equation is called the Feynman-Kac formula (equation (FK) below). This history is summarized succinctly in the Introduction to Barry Simon's admirable book [23], where many additional references are provided.

Since the passage from the parabolic wave equation to the heat equation formally amounts to replacing z by $-iz$ (i.e., in the quantum-mechanical setting, replacing t by $-it$), methods relying on the Feynman-Kac formula are often called imaginary-time methods. A currently flourishing school of

research in quantum field theory is based on this imaginary-time approach ([24]; see also Nelson's book review [25]). A maneuver frequently used by those who think of their work as involving imaginary time is analytic continuation from an imaginary half-axis to a real half-axis. We make no use of imaginary time *per se* or of analytic continuation below. Instead, we begin with the Feynman-Kac formula for the heat equation and, via a trick of Donsker and Varadhan [26], proceed along lines which, as far as we know, are new in the context of mathematical physics.

Wiener integral approaches, some of them merely formal manipulations assuming imaginary time, have been used previously in the theory of wave propagation in random media; see the survey article of Frisch [27] for applications up to 1968. The use of rigorous Wiener integral techniques, rather than Feynman integral methods, affords certain advantages: all of the results of the mathematical theory of measure and integration are automatically at one's disposal. For example, as Chow [28] has pointed out, one may justify the interchange of Wiener integration and expectation in the probability space on which the random fluctuation μ is defined by appealing to Fubini's theorem. In any case, the use of rigorous Wiener integrals holds promise for avoiding the "balancing act, . . . that is, our attempts to draw firm conclusions from not so firm a theory" [19, p.63], which inevitably, it seems, accompanies the use of Feynman integrals. In this spirit, the discussion that follows aims to be rigorous, modulo regularity assumptions (or theorems) about the functions involved.

Consider the inhomogenous heat equation

$$\frac{\partial Y}{\partial z} = \frac{1}{2}\frac{\partial^2 Y}{\partial x^2} - V(x,z)Y \quad , \tag{57}$$

where the unknown function $Y = Y(x,z)$ is subject to the initial condition that $Y(x,0)$ is a prescribed function. Then the Feynman-Kac formula for the solution to (57) is

$$Y(x,z) = E_x\left[\exp\left\{-\int_0^z V(w(s),s)\,ds\right\} Y(w(z),0)\right] , \tag{FK}$$

where $w(s)$ is Brownian motion starting at x, and "E_x" denotes integration (or expectation) in the probability space on which $\{w(s)\}$ is defined, with respect to the corresponding Wiener measure. The right-hand side of (FK) is thus a Wiener integral (or "functional integral", or "path-space integral"). (We note that V is usually assumed to be independent of z; the more general case $V = V(x,z)$ was considered by Faris [29].)

Now suppose that $u = u(x,z)$ is an "arbitrary" function (modulo smoothness assumptions). Then, as Donsker and Varadhan have observed [26, p.19], if we put

$$V(x,z) = \left[\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial z}\right] \Big/ 2u(x,z) , \tag{58}$$

then the solution to (57) is just $Y(x,z) = u(x,z)$. Hence, by (FK), we have

$$u(x,z) = E_x\left[\exp\left\{-\int_0^z \left(\frac{\partial^2 u/\partial x^2 - 2\partial u/\partial z}{2u}\right)(w(s),s)\ ds\right\} u(w(z),0)\right] . \tag{59}$$

Now, suppose also that $u(x,z) = \psi(x,z)$, a solution to the Parabolic Wave Equation (we take k=1 here for convenience, and continue to neglect y-dependence):

$$i\frac{\partial\psi}{\partial z} + \frac{1}{2}\frac{\partial^2\psi}{\partial x^2} - \mu\psi = 0, \quad \mu = \mu(x,z). \tag{60}$$

Then a short computation shows that

$$\left[\frac{\partial^2\psi}{\partial x^2} - 2\frac{\partial\psi}{\partial z}\right] \Big/ 2\psi = -(1+i)\frac{\partial}{\partial z}\left(\log(\psi e^{iM})\right) + i\mu , \tag{61}$$

where

$$M(x,z) \equiv \int_0^z \mu(x,s)\, ds, \tag{62}$$

so that

$$\frac{\partial M}{\partial z} = \mu \, . \tag{63}$$

Substituting (61) into (59), we find

$$\psi(x,z) = E_x\left[\exp\left\{\int_0^z \left[(1+i)\, \frac{\partial}{\partial z}\, \left(\log\left(\psi e^{iM}\right)\right) - i\mu\right](w(s),s)\, ds\right\}\, \psi(w(z),0)\right] \, . \tag{64}$$

In particular, for $\mu \equiv 0$, the solution $\psi_0(x,z)$ of

$$i\, \frac{\partial \psi_o}{\partial z} + \frac{1}{2}\, \frac{\partial^2 \psi_o}{\partial x^2} = 0$$

is just

$$\psi_o(x,z) = E_x\left[\exp\{ \int_o^z \left[(1+i)\frac{\partial}{\partial z}(\log\, \psi_o)\right](w(s),s)\, ds\}\psi_o(w(z),0)\right]. \tag{65}$$

The case of μ a Gaussian random field, independent of x:

In this case we can use the above approach to derive "the usual formula obtained in the Markov approximation" [1, p.898] for the first moment $\langle\psi\rangle$ of ψ. Here and in what follows "$\langle\ \rangle$" denotes expectation in the probability space on which μ is defined.

Since we assume here that $\mu=\mu(z)$, a short calculation shows that

$$\psi_o(x,z) = \psi(x,z)\, e^{iM(z)} \tag{66}$$

(provided that $\psi_o(x,0) = \psi(x,0)$). Thus (64) becomes

$$\psi(x,z) = E_x\left[\exp\{ -i\int_o^z \mu(s)\, ds\}\exp\{\int_o^z[(1+i)\frac{\partial}{\partial z}(\log\psi_o)](w(s),s)\, ds\}\psi_o(w(z),0)\right] \, , \tag{67}$$

or

$$\psi(x,z) = \exp\{-i\int_o^z \mu(s)\, ds\}E_x\left[\exp\{\int_o^z[(1+i)\frac{\partial}{\partial z}(\log\psi_o)](w(s),s)\, ds\}\psi_o(w(z),0)\right],$$

because the factor involving μ is independent of the path $w(s)$. Using (65), we obtain simply

$$\psi\ (x,z) = \exp\{-i\int_0^z \mu(s)\,ds\}\ \psi_0(x,z). \tag{68}$$

We assume that $\langle\mu\rangle = 0$ (so that by (4), $\langle\eta\rangle = 1$, the index of refraction of a homogenous medium). Therefore, $\int_0^z \mu(s)\,ds$ is Gaussian with mean 0. But, if Z is a Gaussian random variable with zero mean, then

$$\left\langle e^{-iZ}\right\rangle = e^{-\frac{1}{2}\langle Z^2\rangle} . \tag{69}$$

Therefore, if we define the function Φ by

$$\Phi(z) \equiv \left\langle\left(\int_0^z \mu(s)\,ds\right)^2\right\rangle^{1/2} , \tag{70}$$

take "<>" of both sides of (68), and use (69) for $Z = \int_0^z \mu(s)\,ds$, we obtain

$$\boxed{\left\langle\psi(x,z)\right\rangle = \psi_0(x,z)e^{-\frac{1}{2}\Phi^2(z)}} , \tag{71}$$

the "usual formula" mentioned above for the first moment of the field ψ from the theory of wave propagation in a random medium. Φ is called the rms phase fluctuation (as computed in first-order geometric optics) and serves as a measure of the strength of the fluctuation μ.

<u>The case of more general μ:</u>

To derive the formula (71) for more general $\mu = \mu(x,z)$, it has become customary to make use of a "Markov approximation." According to Dashen [1, p.896], if

$$\sigma\left(((x-x')^2 + (z-z')^2)^{1/2}\right) \equiv \left\langle\mu(x,z)\ \mu(x',z')\right\rangle \tag{72}$$

is the correlation function of μ, then the Markov approximation is the replacement

$$\sigma\left(((x-x')^2 + (z-z')^2)^{1/2}\right) \to \delta(z-z')\ \hat{\sigma}\ (|x-x'|), \tag{73}$$

where

$$\hat{\sigma}(|x|) = \int_{-\infty}^{\infty} \sigma\left((|x|^2+z^2)^{1/2}\right) dz . \tag{74}$$

In particular, for a Brownian path $w(s)$ in this approximation,

$$\langle \mu(w(s),s)\ \mu\ (w(s'),s')\rangle = \delta(s-s')\ \hat{\sigma}\ (|w(s)-w(s')|) = \delta(s-s')\ \hat{\sigma}\ (0), \tag{75}$$

i.e., μ's at different points of the same path are uncorrelated (hence independent, if Gaussian), and the above expression is independent of $w(s)$ and $w(s')$. Thus, for the path integral representation (64) of μ, equation (75) should amount to ignoring the x-dependence of μ, so that effectively $\mu=\mu(z)$ as above. We shall examine this idea, and the Markov approximation in general, in detail in the sequel.

Important Paths:

Elsewhere [1, p.898], Dashen introduces the Markov approximation as follows: "The parabolic wave equation assumes that the normals to the wave fronts point in directions that are close to the z-axis. In terms of the path integral this means that for the important paths. . .

$$(w(z)-w(z'))^2 + (z-z')^2 \cong (z-z')^2 \tag{76}$$

[in our notation for paths]." It is not hard to see that (76) and (73) are essentially equivalent. However, here the approximation is explicitly made only for "important paths." It is clear from Dashen's quoted explanation that he has in mind the physically important rays in situations where the small-angle approximation leading to the Parabolic Wave Equation is valid. But there seems to be no clear mathematical basis for neglecting paths that wander away from the z-axis in Dashen's Feynman integrals: the "measure" in path space corresponding to Feynman integrals, if we could construct one, would be infinite-dimensional Lebesque measure, and therefore translation invariant. Much the same criticism

would seem to apply to the Brownian paths in our Wiener integral formula (64).

In order to make the Markov approximation for "important paths" more nearly exact, we should use a measure in path space such that paths which wander far from the z-axis are truly negligible, i.e. constitute a set of negligibly small measure. An ideal candidate for such a measure would seem to be the measure corresponding to the random ray process of section III. Assuming that the parabolic approximation is self-consistent, the quantity $|\psi(x,z)|^2$, which is proportional to the acoustic intensity, should be concentrated near the z-axis. But $|\psi(x,z)|^2 = \rho(x,z)$, the probability density of the position $\mathbf{x}(z)$ of the random ray at range z. Thus, in the sequel we shall use the Cameron-Martin-Girsanov formula ([12]; see also [30, chapter 7], where this name is not used) to change variables in the path integral of equation (64) from the Brownian path w(s) to the random ray x(s).

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